

# CYCLES FOR ASYMPTOTIC SOLUTIONS AND WEYL GROUP

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ABSTRACT. In this paper cycles for asymptotic solutions are described. Cycles are enumerated by elements of symmetric group. Leading asymptotic and leading coefficient are calculated. Value of certain multiple integral over special cycles is calculated with the help of result of Opdam.

## 0. INTRODUCTION

In this paper cycles for asymptotic solutions are described. Cycles are enumerated by elements of symmetric group. Leading asymptotic and leading coefficient are calculated. Value of certain multiple integral over special cycles is calculated with the help of result of Opdam.

In [33] homological treatment of Harish-Chandra decomposition is given. We start with recalling the necessary material from [18, 19]. We restrict ourselves to the case of root system of type  $A_n$ . For better exposition the reader should consult directly with the papers [18, 19, 38, 39, 40].

**0.1. Differential operator of second order.** Let  $L$  be the following differential operator

$$L = \sum_{i=1}^{n+1} (z_i \frac{\partial}{\partial z_i})^2 - k \sum_{i < j} \frac{z_j + z_i}{z_j - z_i} (z_i \frac{\partial}{\partial z_i} - z_j \frac{\partial}{\partial z_j}).$$

Let  $u_i = \log(z_i)$  for  $i = 1, 2, \dots, n+1$ . In coordinates  $u_1, \dots, u_{n+1}$  operator  $L$  reads as:

$$L = \sum_{i=1}^{n+1} \frac{\partial^2}{\partial u_i^2} - k \sum_{i < j} \frac{1 + e^{u_i - u_j}}{1 - e^{u_i - u_j}} (\frac{\partial}{\partial u_i} - \frac{\partial}{\partial u_j})$$

*Remark.* In notations of Harish-Chandra cf.[19] operator  $L$  reads as:

$$L = (H_1^2 + \dots + H_l^2) + \sum_{\alpha \in P_+} \frac{(e^\alpha + e^{-\alpha})}{(e^\alpha - e^{-\alpha})} H_\alpha.$$

Operator  $L$  plays the crucial role in the theory of zonal spherical functions on noncompact Riemannian spaces  $G/K$ .

We, following [18], use  $\alpha$  twice as usual restricted roots and  $k$  half the usual multiplicities.

The theory of zonal spherical functions goes back to H.Weyl and E.Cartan [ 42 ]. They used mostly integral methods. In [41 ] I.Gelfand suggested to use Laplace-Casimir operators , cf. also [26 ] .

$L$  is the radial part of Laplace-Casimir operator of second order with respect to Cartan decomposition ( $G = KAK$ ).

## 0.2. Root system of type $A_n$ .

Let  $V$  be a Euclidean  $(n+1)$ -dimensional space with inner product  $(.,.)$ . For  $\alpha \in V$

$$r_\alpha(\lambda) = \lambda - (\lambda, \alpha^\vee) \alpha$$

$$(\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)})$$

denotes the orthogonal reflection with respect to the hyperplane  $(\alpha, .) = 0$ .

Let

$$e_1, e_2, \dots, e_{n+1}$$

be the orthonormal basis of  $V$ . Let  $E$  be a  $n$ -dimensional subspace of  $V$ , which is orthogonal to  $e_1 + e_2 + \dots + e_{n+1}$  :

$$E = \{(u_1, u_2, \dots, u_{n+1}); \quad u_1 + u_2 + \dots + u_{n+1} = 0\}.$$

Then  $E$  is also Euclidean and we will sometimes identify it with its dual without additional comments. Then the set  $R$  of  $n(n+1)$  vectors in  $E$ :

$$R := \{e_i - e_j \mid i \neq j\}$$

is called **the root system of type  $A_n$** .

Take a vector  $\alpha \in R$ , say  $\alpha = e_i - e_j$ . Then reflection  $r_\alpha$  permutes  $i$ th and  $j$ th coordinate:

$$\begin{aligned} r_\alpha(\lambda_1, \lambda_2, \dots, \lambda_i, \dots, \lambda_j, \dots, \lambda_{n+1}) \\ = (\lambda_1, \lambda_2, \dots, \lambda_j, \dots, \lambda_i, \dots, \lambda_{n+1}) \end{aligned}$$

The Weyl group  $W$  is the subgroup of isometries of  $E$ , which is generated by reflections  $r_\alpha$ ,  $\alpha \in R$ . Obviously, for the root system of type  $A_n$   $W$  coincides with the group of permutation of  $n+1$  letters  $S_{n+1}$ . The set of vectors  $R_+ = \{e_i - e_j \mid i < j\}$  is the usual choice for the positive roots. The halfsum of positive roots is equal to:

$$\delta = \frac{1}{2} \sum_{\alpha \in R_+} \alpha = \left( \frac{n}{2}, \frac{n-2}{2}, \dots, \frac{-n}{2} \right)$$

Set

$$\rho = \rho(k) = \frac{1}{2}k \sum_{\alpha \in R_+} \alpha = k\delta = \left( \frac{n}{2}k, \frac{n-2}{2}k, \dots, \frac{-n}{2}k \right) .$$

Let  $\alpha_i = e_i - e_{i+1}$ ,  $i = 1, 2, \dots, n$ . Then vectors  $\alpha_1, \alpha_2, \dots, \alpha_n$  are called simple roots, and any positive root is expressed as a linear combination of simple roots with nonnegative coefficients. Introduce a partial ordering on  $E$  by

$$\lambda \geq \mu \text{ if and only if } \lambda - \mu = \sum_{j=1}^n l_j \alpha_j \text{ with } l_j \in \mathbb{Z}_+, \text{ where } \mathbb{Z}_+ = 0, 1, 2, \dots$$

Let  $\Lambda_1, \Lambda_2, \dots, \Lambda_n$  be fundamental weights, i.e.  $\Lambda_i \in E$ ,  $(\Lambda_i, \alpha_j) = \delta_{ij}$ , where  $\delta_{ij}$  is Kronecker's delta.

**0.3. Ring of differential operators .** Let  $\mathcal{R}$  be the algebra of functions generated by the functions

$$\left\{ \frac{1}{1 - e^{u_i - u_j}}, i < j \right\}$$

Consider  $\mathcal{D}$  the set of all differential operators with coefficients in  $\mathcal{R}$ , i.e. of the form

$$\sum_i f_i(u_1, \dots, u_{n+1}) P_i \left( \frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_{n+1}} \right),$$

where  $f_i(u_1, \dots, u_{n+1}) \in \mathcal{R}$ . Using the identity

$$\frac{1}{1 - e^{u_i - u_j}} = \sum_{m=0}^{\infty} e^{m(u_i - u_j)}$$

For each differential operator from  $\mathcal{D}$  one obtains a representation of the form:

$$P = \sum_{\mu \geq 0} e^{\mu(u)} p_{\mu} \left( \frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_{n+1}} \right),$$

where  $\mu(u) = (\mu, u)$  and  $u = (u_1, \dots, u_{n+1})$

In particular,  $L$  is a differential operator of  $\mathcal{D}$  and  $L$  has the following asymptotic expansion:

$$L = \sum \frac{\partial^2}{\partial u_i^2} - 2 \sum_i (\rho, e_i) \frac{\partial}{\partial u_i} - 2 \sum_{i < j} k \sum_{m=1}^{\infty} e^{m(u_i - u_j)} \left( \frac{\partial}{\partial u_i} - \frac{\partial}{\partial u_j} \right).$$

**0.4. Definition.** (Harish-Chandra homomorphism) Harish-Chandra homomorphism is defined as the algebra homomorphism:  $\gamma = \gamma(k): \mathcal{D} \rightarrow \mathbb{C}[E \otimes_{\mathbb{R}} \mathbb{C}]$  by

$$\gamma: P = \sum_{\mu \geq 0} e^{\mu(u)} p_{\mu} \left( \frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_{n+1}} \right) \rightarrow \{\lambda \rightarrow p_0(\lambda + \rho)\},$$

where  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{n+1})$  such that  $\lambda_1 + \lambda_2 + \dots + \lambda_{n+1} = 0$  and  $\rho$  is defined as above.

In particular,

$$\gamma(L)(\lambda) = (\lambda + \rho, \lambda + \rho) - (2\rho, \lambda + \rho) = (\lambda, \lambda) - (\rho, \rho).$$

Also, one should notice, that

$$\gamma(L)(w\lambda) = (w\lambda + \rho, w\lambda + \rho) - (2\rho, w\lambda + \rho) = (w\lambda, w\lambda) - (\rho, \rho) = (\lambda, \lambda) - (\rho, \rho)$$

**0.5. Definition.** (Action of the Weyl group  $W$  on  $\mathcal{R}$  and  $\mathcal{D}$ )

The Weyl group acts on  $\mathcal{R}$  by

$$w: \frac{1}{1 - e^{u_i - u_j}} \rightarrow \frac{1}{1 - e^{u_{w(i)} - u_{w(j)}}}.$$

One should notice that

$$(1 - e^{u_j - u_i})^{-1} = 1 - (1 - e^{u_i - u_j})^{-1}$$

and thus the action of the Weyl group is defined correctly.

The Weyl group acts on differential operators from  $\mathcal{D}$  as

$$\begin{aligned} w: \sum_i f_i(u_1, \dots, u_{n+1}) P_i \left( \frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_{n+1}} \right) \\ \rightarrow \sum_i f_i(u_{w(1)}, \dots, u_{w(n+1)}) P_i \left( \frac{\partial}{\partial u_{w(1)}}, \dots, \frac{\partial}{\partial u_{w(n+1)}} \right) \end{aligned}$$

In particular,  $L$  is Weyl group invariant differential operator.

**0.6. Symmetric polynomials.** Consider symmetric polynomials in  $u_1, \dots, u_{n+1}$ . Let  $\sigma_1 = \sum_{i=1}^{n+1} u_i$ ,  $\sigma_2 = \sum_{i < j} u_i u_j$ ,  $\dots$ ,  $\sigma_{n+1} = \prod_{i=1}^{n+1} u_i$  be elementary symmetric polynomials.

**0.7. Theorem.** *Any symmetric polynomial  $s = s(u_1, \dots, u_{n+1})$  over  $\mathbb{C}$  can be represented as a polynomial over  $\mathbb{C}$  of elementary symmetric polynomials  $\sigma_1, \dots, \sigma_{n+1}$ :*

$$s = s(\sigma_1, \dots, \sigma_{n+1})$$

and there is no algebraic relations between elementary symmetric polynomials  $\sigma_i$ .

**0.8. Algebra of commuting differential operators.** Set  $\mathbb{D} = \mathbb{D}(k)$  for the algebra of all Weyl group invariant differential operators in  $\mathcal{D}$  which commute with operator  $L = L(k)$ :

$$\mathbb{D} = \{P \in \mathcal{D} \mid [L, P] = 0, w(P) = P \text{ for any } w \in W\}$$

*Remark.* Restriction of Harish-Chandra homomorphism  $\gamma$  from  $\mathcal{D}$  to  $\mathbb{D}$  will be also called Harish-Chandra homomorphism.

**0.9. Theorem.** 1). *Differential operator  $P = \sum_{\mu \geq 0} e^{\mu(u)} p_\mu(\frac{\partial}{\partial u})$  in  $\mathcal{D}$  commutes with  $L$ :  $[L, P] = L \circ P - P \circ L = 0$  if and only if polynomials  $p_\mu(\lambda)$  satisfy the recurrence relations:*

$$(2\lambda - 2\rho + \mu, \mu) p_\mu(\lambda) = 2k \sum_{\alpha \in R_+} \sum_{j=1}^{\infty} \{(\lambda + \mu - j\alpha, \alpha) p_{\mu-j\alpha}(\lambda) - (\lambda, \alpha) p_{\mu-j\alpha}(\lambda + j\alpha)\}.$$

2). *For any symmetric polynomial  $\sigma = \sigma(\lambda)$  set  $p_0(\lambda) = \sigma(\lambda - \rho)$ . Then all the recurrence relations above can be solved and obtained differential operator is in  $\mathcal{D}$ , Weyl group invariant and thus in  $\mathbb{D}$ . Furthermore, all elements of  $\mathbb{D}$  are obtained in this way, or in other words, Harish-Chandra homomorphism  $\gamma = \gamma(k): \mathbb{D} \rightarrow \mathbb{C}[E \otimes_{\mathbb{R}} \mathbb{C}]^W$  is surjective.*

3). *All differential operators constructed in 2). commute not only with  $L$ , but as well with each other.*

For example, to 1 we assign in this way trivial differential operator, namely, identity, to  $\sigma_1 = \sum \lambda_i$  we assign  $\sum \frac{\partial}{\partial u_i}$ , to  $\sum \lambda_i^2$  we assign  $L + (\rho, \rho)$ , and so on.

*Remark 0.10.* For explicit form of generators of  $\mathbb{D}(k)$  reader should consult with [38,39,40].

### Hypergeometric system of differential equations.

**0.11. Definition.** ( Hypergeometric system of differential equations)  
The system of differential equations

$$P\phi = \gamma(P)(\lambda)\phi \quad P \in \mathbb{D}, \quad \lambda = (\lambda_1, \dots, \lambda_{n+1}) \text{ s.t. } \lambda_1 + \dots + \lambda_{n+1} = 0$$

is called the system of hypergeometric (partial) differential equations.

One may consider this system in variables  $u_i$  and in variables  $z_i$ . To obtain a system in  $z_i$  one should replace  $\frac{\partial}{\partial u_i}$  by  $z_i \frac{\partial}{\partial z_i}$  and  $(1 - e^{u_i - u_j})^{-1}$  by  $\frac{z_j}{z_j - z_i}$  correspondingly.

**0.12. Definition.** Set  $H^{reg}$  as a  $(n+1)$ -dimensional complex plane  $\mathbb{C}^{n+1} = \{(z_1, \dots, z_{n+1})\}$  with diagonals  $\{z_i = z_j\}$ ,  $i < j$ , and with coordinate hyperlanes  $\{z_i = 0\}$  being deleted:

$$H^{reg} = \mathbb{C}^{n+1} \setminus \left( \bigcup \{z_i = z_j\} \bigcup \{z_i = 0\} \right)$$

*Remark.* This configuration is also called an affine configuration. It became classical now and was studied by Arnold, Brieskorn, Orlik, Solomon, Schechtman, Varchenko,...

**0.13. Theorem.** (Holonomicity on  $H^{reg}$ ) Locally on  $H^{reg}$  solution space of the system of hypergeometric equations has dimension equal to the order  $|W|$  of the Weyl group  $W$  and consists of analytic functions.

**0.14. Asymptotic solutions.** Operator  $L$  permits the following asymptotic expansion :

$$L = \sum \frac{\partial^2}{\partial u_i^2} - 2 \sum_i (\rho, e_i) \frac{\partial}{\partial u_i} - 2 \sum_{i < j} k \sum_{m=1}^{\infty} e^{m(u_i - u_j)} \left( \frac{\partial}{\partial u_i} - \frac{\partial}{\partial u_j} \right).$$

Consider solutions of

$$L\phi = ((\lambda, \lambda) - (\rho, \rho))\phi$$

of the form

$$\phi = \phi(\mu, k, e^u) = \sum_{\nu \geq \mu} \Gamma_\nu(\mu, k) e^{\nu(u)} \quad ,$$

where  $\mu = w\lambda + \rho$ .

Then coefficients  $\Gamma_\nu(\mu, k)$  satisfy to Freudenthal-type recurrence relations

$$\{(\nu - \rho, \nu - \rho) - (\mu - \rho, \mu - \rho)\} \Gamma_\nu(\mu, k) = 2 \sum_{\alpha \in R_+} k \sum_{j=1}^{\infty} (\nu - j\alpha, \alpha) \Gamma_{\nu - j\alpha}(\mu, k).$$

For generic  $\lambda$  ( $(\lambda, \alpha^\vee) \notin \mathbb{Z}$ ,  $\alpha \in R$ ) solutions  $\phi(w\lambda + \rho, k, e^u)$  are linearly independent and provide a basis of linear space of solutions to the whole system of hypergeometric equations.

These solutions  $\phi(w\lambda + \rho, k, e^u)$  (or  $\phi(w\lambda + \rho, k, z)$ ) are called **asymptotic solutions**.

*So instead of dealing with the whole hypergeometric system, one can deal with differential operator of second order and asymptotic solutions only.*

### 0.15. Integral representation for the solutions to the hypergeometric system of differential equations.

A. Matsuo [15] proved that the hypergeometric system of differential equations in the case of root system of type  $A_n$  (which is only being considered in this paper) is related to the Knizhnik-Zamolodchikov equation in conformal field theory, cf. also [16]. In particular this implies that solutions of the system are represented as certain multidimensional integrals cf. [1].

Namely, take integration variables indexed as  $\{t_l^{(j)}; j = 1, \dots, n, l = j, \dots, n\}$ . Take a total ordering  $<$  in the set of indices  $I$ .

$$\begin{aligned} \omega(z, t) = & \prod_{(j,l) \in I} e^{-\frac{k}{2}(\lambda, \alpha_j) t_l^{(j)}} \prod_{1 \leq i \leq n+1} e^{\frac{k}{2}(\lambda, \Lambda_1) z_i} \\ & \times \prod_{1 \leq i \leq n+1} \prod_{1 \leq l \leq n} \left( \sinh \frac{z_i - t_l^{(1)}}{2} \right)^{-k} \prod_{(j,l) < (j',l')} \left( \sinh \frac{t_l^{(j)} - t_{l'}^{(j')}}{2} \right)^{k(\alpha_j, \alpha'_{j'})} \end{aligned}$$

Then

$$\int_{\Gamma} \omega(z, t) \phi(z, t) dt_1^{(1)} \dots dt_n^{(n)}$$

where  $\phi(z, t)$  is a polynomial of  $\coth \frac{t_l^{(j)} - z_i}{2}$  and  $\coth \frac{t_l^{(j)} - t_{l'}^{(j')}}{2}$  written in a complicated manner in [1] for a certain choice of  $\Gamma$  provide solutions for hypergeometric system of differential equations. Contours for asymptotic solutions and for zonal spherical function itself are not described in [15].

In this paper we provide another integral representation for the solutions cf. theorem 6.3. Namely, we use

$$\begin{aligned} \omega(z, t) = & \prod (z_i - z_j)^{1-2k} \\ & \times \prod_{(j,l) \in I} t_l^{(j)(\lambda+\rho, -\alpha_j)} \prod_{1 \leq i \leq n+1} z_i^{(\lambda+\rho, e_1)} \\ & \times \prod_{1 \leq i \leq n+1} \prod_{1 \leq l \leq n} (z_i - t_l^{(1)})^{k-1} \prod_{(j,l) < (j',l')} (t_l^{(j)} - t_{l'}^{(j')})^{(1-k)(\alpha_j, \alpha'_j)} \end{aligned}$$

Let's imbed  $V$  into  $\mathbb{R} \times V$  as  $v \rightarrow (0, v)$ . Let  $\alpha_0 = e_0 - e_1$ . Then our form  $\omega(z, t)$  can be written as:

$$\begin{aligned} \omega(z, t) = & \prod_{i < j} (z_i - z_j)^{-1} \prod (z_i - z_j)^{(1-k)(\alpha_0, \alpha_0)} \\ & \times \prod_{(j,l) \in I} t_l^{(j)(\lambda+\rho, -\alpha_j)} \prod_{1 \leq i \leq n+1} z_i^{(\lambda+\rho, -\alpha_0)} \\ & \times \prod_{1 \leq i \leq n+1} \prod_{1 \leq l \leq n} (z_i - t_l^{(1)})^{(1-k)(\alpha_0, \alpha_1)} \prod_{(j,l) < (j',l')} (t_l^{(j)} - t_{l'}^{(j')})^{(1-k)(\alpha_j, \alpha'_j)} \end{aligned}$$

Solutions to the hypergeometric system are given by the integral

$$\int \omega(z, t) dt_1^{(1)} \dots dt_n^{(n)}$$

over appropriate contour of integration. Here our  $z_i$  is  $e^{z_i}$  of Matsuo.

In the paper we use another indexation of variables of integration  $\{t_{i,j}; i = 1, \dots, j, j = 1, \dots, n\}$ . Variables  $t_{i,j}$  have a nice geometric origin in elliptic coordinates cf.[25], also they are used in [25] in the Plancherel theorem.

*Remark 0.16.* We obtained the form in the following way. Zonal spherical function is defined as  $\phi(g) = (T_g \xi, \xi)$ , where  $T_g$  is a unitary representation of  $g \in G$ ,  $\xi$  is invariant vector of maximal compact subgroup cf.[25]. The zonal spherical function for  $SL(n, \mathbb{C})$  was calculated in [25] and using the same methods for  $SL(n, \mathbb{R})$  in [35]. Essentially, the so-called elliptic coordinates are used. The case of  $SL(n, \mathbb{C})$  corresponds to  $k = 1$  in [18] and case of  $SL(n, \mathbb{R})$  corresponds to  $k = \frac{1}{2}$ . So if one considers integral representations for zonal spherical functions for  $k = 1$  and for  $k = \frac{1}{2}$  (one should also replace  $\delta_i^2$  by  $z_i$  and  $\frac{i\lambda}{2}$  by  $\lambda$ , i.e. adopt normalizations of [18]) and then connects powers of factors linearly on  $k$ , one obtains exactly the above formulas. Also, in calculation of zonal spherical function integration is taken over a distinguished cycle. Its role is discussed in [33].

### 0.17. Organization of the paper.

We describe contours for integration for asymptotic solutions  $\phi(w\lambda + \rho, k, z)$  and enumerate them by the elements of symmetric group. Namely, for each element  $w \in S_n$  we put into correspondence a diagram, cf. sections 1,2.

For the diagrams we borrow very convenient graphical notation of [2]. We also discuss the interrelation of diagrams and Gelfand-Zetlin patterns in section 3 (Weyl group orbit of the lowest weight is described with the help of diagrams).

The system of contours  $\Delta_w = \Delta_w(z)$  for integration for each diagram (and thus for each element  $w \in S_n$ ) is described in section 4, cf. definition 4.3.

Section 5 is devoted to description of the choice of the phase of multi-valued form  $\omega$  over cycle  $\Delta_w$ .

In section 6 we calculate the leading asymptotics ( $w\lambda + \rho$ ) of the integral and leading coefficient, and prove that integrals satisfy to second order differential equation  $L\phi = ((\lambda, \lambda) - (\rho, \rho))\phi$  and thus to the whole system of hypergeometric equations. Finally (theorem 6.8), we calculate the value of the integral over described cycles when all the arguments collapse to the unity. We use result of [21], and it is consistent with [17, 33].

*Remark 0.18.* In the case of  $k = 1$  our integral simplifies dramatically (boils down to integrals considered in [25]), namely only affine part (or Mellin part) of the integral survives. Also, the zonal spherical function in this case is equal to the quotient of some determinant by  $\prod (z_i - z_j)$ . The determinant is a sum of  $n!$  terms with the signs plus or minus. So one can try to select each term from the determinant, i.e. to find an appropriate domain of integration. This can be easily done. Modification of these domains to the case of generic  $k$  is done in the definition 4.3.

*Remark 0.19.* Our integral representation for the solutions  $\int \omega(z, t)dt$  is in well agreement with the formula of Harish-Chandra for zonal spherical functions:

$$\phi_\lambda(g) = \int e^{(i\lambda + \rho)(A(kg))} dk, \quad g \in G$$

For more details about formula of Harish-Chandra and application of the stationary phase method to the formula cf. [20] and [43], correspondingly.

## 1. DIAGRAMS

The notion of a diagram was introduced in [1] in the context of Knizhnik-Zamolodchikov equation, and later similar notion was introduced in [2]

in the context of multidimensional determinants. We will use the notion of a diagram in the form of [2], in particular, we borrow the graphical notation of [2].

Compare also the diagrams with sequences Seq of [3].

Fix some positive integer  $n$ . Consider the set of  $\frac{n(n+1)}{2}$  points, indexed by pairs of integers  $\{(i, j) \mid i = 1, \dots, j, j = 1, \dots, n\}$ . It is helpful to organize the points in the form of a pattern, so that points are divided in  $n$  rows,  $j$ th row is formed by points  $\{(i, j) \mid i = 1, \dots, j\}$ ; point  $(i, j)$  is located under and between points  $(i, j+1)$  and  $(i+1, j+1)$  (fig. 1a).

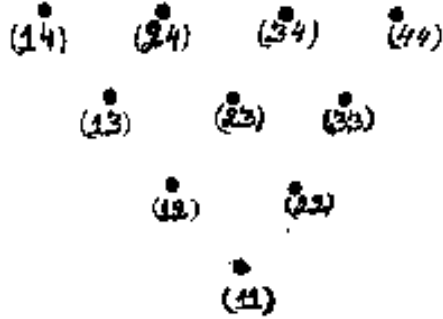


FIGURE 1A.  $n = 4$ . Points  $(i, j)$  organized in a pattern;  $j$  is the number of the row,  $i$  is the number in the row

Now mark with a cross one point in each row. Let  $\{(i_j, j)\}$  be the subset of marked points (fig. 1b).

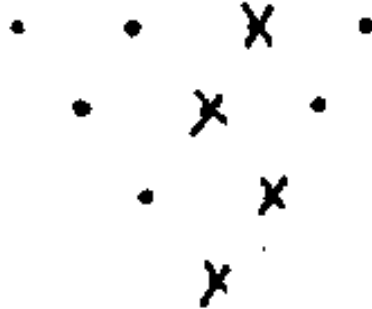


FIGURE 1B. One point is marked in each row.

Finally, draw an arrow for each point  $(i, j)$  with the source in this point

$(i, j)$  and target  $tar(i, j)$  in the next  $j + 1$ th row defined as:

$$tar(i, j) = \begin{cases} (i, j + 1), & \text{if } i < i_{j+1} \\ (i + 1, j + 1), & \text{if } i \geq i_{j+1} \end{cases}^1$$

Note: neither arrow has a marked point as its target.  
In this way one obtains fig 1c.

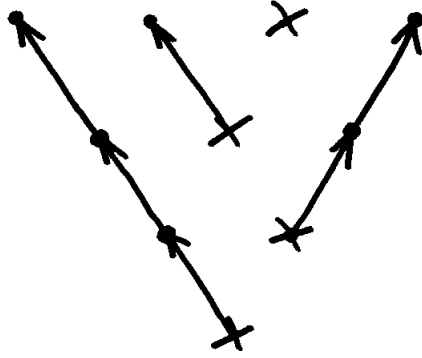


FIGURE 1c. Example of a diagram.

**Definition 1.1.** A triple

$$(\{i, j\}, \{i_j, j\}, tar)$$

consisting of:

set of points  $\{(i, j) \mid i = 1, \dots, j; j = 1, \dots, n\}$ ,

set of marked points  $\{(i_j, j) \mid j = 1, \dots, n\}$

and function  $tar$  defined above will be called **a diagram**.

*Remark 1.2.* One can see that a diagram is determined by the set of marked points.

One easily notices that there are exactly  $n!$  different diagrams with  $n$  rows. So it is desirable to enumerate diagrams by elements of Symmetric group  $S_n$ . Calculations connected with the integral led us to the following propositions and theorems. One may consider them as a peculiar show up of quantum groups.

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<sup>1</sup>Recall that  $\{(i_j, j)\}$  is the set of marked points and  $(i_{j+1}, j + 1)$  is the only marked point in  $j + 1$ th row.

## 2. ENUMERATION OF DIAGRAMS BY ELEMENTS OF SYMMETRIC GROUP

**Definition 2.1.** Consider a diagram. For a point  $(i, j)$ , provided  $(i, j)$  is not a marked point,  $j = 2, \dots, n$ , define its **source** as  $(i_1, j - 1)$ , s.t.  $\text{tar}(i_1, j - 1) = (i, j)$ , i.e.  $\text{source} = \text{tar}^{-1}$ , which is defined on the set of nonmarked points.

Consider a diagram with  $n$  rows and a point  $(i, n)$  in the  $n$ th row. Consider also along with  $(i, n)$  its  $\text{source}(i, n)$ ,  $\text{source}(\text{source}(i, n))$ , and so on until we get a marked point. Say,  $\text{source}^{w(i)-1}(i, n)$  is a marked point.

**Proposition 2.2.** *Correspondence  $i \rightarrow w(i)$  correctly defines permutation of numbers  $1, \dots, n$ , i.e. an element of symmetric group  $S_n$ . This is a one-to-one correspondence between diagrams with  $n$  rows and symmetric group  $S_n$ .*

*Remark 2.3.* If  $(i_n, n)$  is marked point in  $n$ th row of a diagram with  $n$  rows, then  $w(i_n) = 1$ . Also if  $(i_{n-1}, n - 1)$  is a marked point in  $n - 1$ th row, then  $w(i_{n-1}) = 2$ , provided  $i_{n-1} < i_n$  and  $w(i_{n-1} + 1) = 2$ , provided  $i_{n-1} \geq i_n$  and so on.

*Remark 2.4.* In other words one can describe the correspondence between diagrams and elements of Symmetric group as follows. Consider a diagram as an oriented graph and forget orientation. For  $i = 1, \dots, n$  define  $w(i)$  as the number of points in the connected component of the point  $(i, n)$ .

Symmetric group  $S_n$  has standard generators  $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ , where  $\sigma_i$  permutes  $i$  and  $i + 1$ .

**Definition 2.5.** The length  $l(w)$  of an element  $w \in S_n$  is the minimal integer  $p \geq 0$ , s.t.  $w$  admits a presentation

$$w = \sigma_{i_1} \sigma_{i_2} \dots \sigma_{i_p}.$$

Any presentation of  $w$  as a product of  $p = l(w)$  generators is called a reduced presentation.

**Theorem 2.6.** *Let a diagram*

$$(\{(i, j) \mid i = 1, \dots, j, j = 1, \dots, n\}, \{(i_j, j) \mid j = 1, \dots, n\}, \text{tar})$$

*corresponds to an element  $w \in S_n$ . Then the length  $l(w)$  of an element  $w$  is equal to:*

$$l(w) = \sum_{j=1}^n (i_j - 1)$$

*Remark 2.7.* Arrow with the source  $t_{ij}$  has either  $t_{i,j+1}$  or  $t_{i+1,j+1}$  as its target. So one can say that the arrow is **to the left** or **to the right**. Hence the theorem 2.6 says that  $l(w)$  is equal to the number of arrows, which are to the left.

The following nice theorem is well known, cf. [4].

**Theorem 2.8.** *For symmetric group  $S_n$  the following identity holds:*

$$\sum_{w \in S_n} q^{l(w)} = \frac{(1-q)}{(1-q)} \frac{(1-q^2)}{(1-q)} \cdots \frac{(1-q^n)}{(1-q)} .$$

Though throughout this paper we mainly using Symmetric group  $S_n$  for purposes of dealing with diagrams, for a moment we would like to use the diagrams to obtain an extension of the above theorem.

**Theorem 2.9.** *(Multiparametric deformation of symmetric group) Denote by  $\{i_j(w), j\}$  set of marked points of diagram corresponding to  $w \in S_n$ . Then the following identity holds:*

$$\sum_{w \in S_n} \prod_{j=1}^n q_j^{i_j(w)-1} = \frac{(1-q_1)}{(1-q_1)} \frac{(1-q_2^2)}{(1-q_2)} \cdots \frac{(1-q_n^n)}{(1-q_n)} .$$

This theorem is an easy application of the formula for the sum of geometric progression

$$1 + x + x^2 + x^3 + \dots + x^{n-1} = \frac{1 - x^n}{1 - x} .$$

**Theorem 2.10.** *Suppose  $w$  corresponds to a diagram with set of marked points  $\{(i_j, j)\}$ . Then  $w$  admits a presentation:*

$$w = w_n w_{n-1} \dots w_2 w_1 ,$$

where

$$w_k = \begin{cases} \sigma_k \sigma_{k+1} \dots \sigma_{i_{n-k+1}+k-2}, & \text{if } i_{n-k+1} > 1 \\ id, & \text{if } i_{n-k+1} = 1 \end{cases}$$

Moreover, presentation of  $w$  in terms of generators  $\sigma_i$ ,  $i = 1, \dots, n-1$ , naturally obtained from the above presentation by expanding  $w_k$  in terms of generators (omitting first those  $w_k$  which are identities) is a reduced presentation of  $w$ , in particular, it contains exactly  $\sum_{j=1}^n (i_j - 1) = l(w)$  factors.

**2.11. Proof of theorem 2.6.** Theorem 2.10 provides us for each  $w \in S_n$  a presentation with  $\sum_{j=1}^n (i_j(w) - 1)$  factors. Consequently, for each  $w$   $\sum_{j=1}^n (i_j(w) - 1) \leq l(w)$ . But comparison of theorems 2.8 and 2.9 shows that all inequalities are in fact equalities. Thus theorem 2.6 is proved.

**Definition 2.12.** (Partial ordering on Symmetric Group) Consider two elements of Symmetric Group  $S_n$ , say,  $w_1$  and  $w_2$ . Let diagram corresponding to  $w_1$  ( $w_2$ ) has  $\{i_j(w_1), j\}$  ( $\{i_j(w_2), j\}$ ) as the set of marked points. We say that  $w_1$  **is less than or equal to**  $w_2$ :  $w_1 \leq w_2$  if for each  $j = 1, \dots, n$

$$i_j(w_1) \leq i_j(w_2).$$

**Proposition 2.13.**

a). For each element  $w$  of Symmetric group  $S_n$  ( $w \in S_n$ ) there are exactly

$$\prod_{j=1}^n (j - i_j(w) + 1)$$

elements which are greater than or equal to  $w$ .

b).

$$\sum_{w' \geq w} q^{l(w')} = q^{l(w)} \prod_{j=1}^n \frac{1 - q^{j - i_j(w) + 1}}{1 - q}$$

c). For each element  $w$  of Symmetric group  $S_n$  ( $w \in S_n$ ) there are exactly

$$\prod_{j=1}^n i_j(w)$$

elements which are less than or equal to  $w$ .

d).

$$\sum_{w' \leq w} q^{l(w')} = \prod_{j=1}^n \frac{1 - q^{i_j(w)}}{1 - q}$$

e). If  $w \leq w'$ , then  $l(w) \leq l(w')$ .

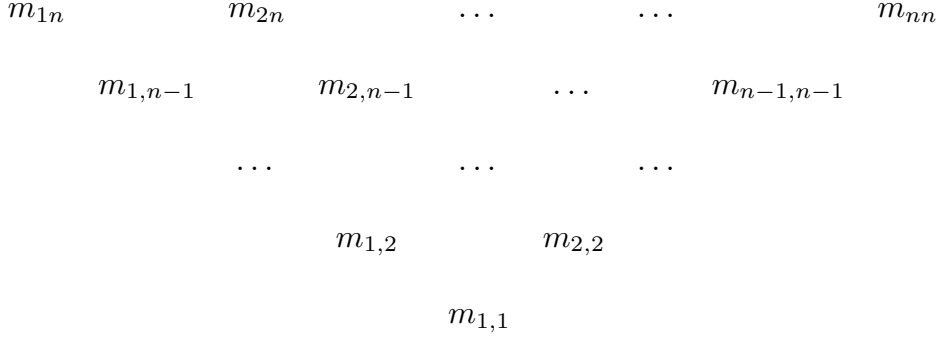


FIGURE 2. Gelfand-Zetlin pattern

### 3. GELFAND-ZETLIN PATTERNS

In [5] finite-dimensional representation of  $gl(n)$  with highest weight  $(m_n, m_{n-1}, \dots, m_1)$  s.t.  $m_n \geq m_{n-1} \geq \dots \geq m_1$  is proved to have a nice basis, which elements are enumerated by the so-called Gelfand-Zetlin patterns, i.e. set of numbers  $m_{pq}$ ,  $p \leq q$ ,  $q = 1, \dots, n-1$  arranged in the following pattern:

The numbers are arbitrary integers which satisfy inequalities  $m_{p,q+1} \leq m_{pq} \leq m_{p+1,q+1}$ ,  $p = 1, \dots, q$ ,  $q = 1, \dots, n-1$ . We changed the usual inequalities to the opposite. The numbers  $m_1, \dots, m_i, \dots, m_n$  which define the representation are denoted by  $m_{in}$  and placed in the  $n$ th row.

Consider a diagram  $\{(i, j), (i_j, j), \text{tar}\}$  with  $n$  rows, which corresponds to an element  $w \in S_n$ .

For the highest vector  $m_n \geq m_{n-1} \geq \dots \geq m_1$  and  $w \in S_n$  we put into correspondence a Gelfand-Zetlin pattern, which is uniquely defined by the relations :

$$m_{in} = m_n, \quad m_{pq} = m_{\text{tar}(p,q)}.$$

**Theorem 3.1.** *The above correspondence  $w \rightarrow \{m_{pq}(w)\}$  is actually the action of  $w \in S_n$  on the lowest weight  $(m_1, m_2, \dots, m_n)$  of representation of  $gl(n)$ , i.e.  $\{m_{pq}(w)\}$  is the only vector of weight  $(m_{w(1)}, m_{w(2)}, \dots, m_{w(n)})$ .*

*Remark 3.2.* This section is aimed to emphasize the philosophical relation between Harish-Chandra decomposition and BGG resolution.

## 4. CYCLES FOR ASYMPTOTIC SOLUTIONS

**Definition 4.1.** By a **bump function** we mean a  $C^1$  function

$$f_\epsilon: [0, 1] \rightarrow [0, \epsilon],$$

such that

$$f_\epsilon(x) = 0 \iff x = 0 \text{ or } x = 1 \quad .$$

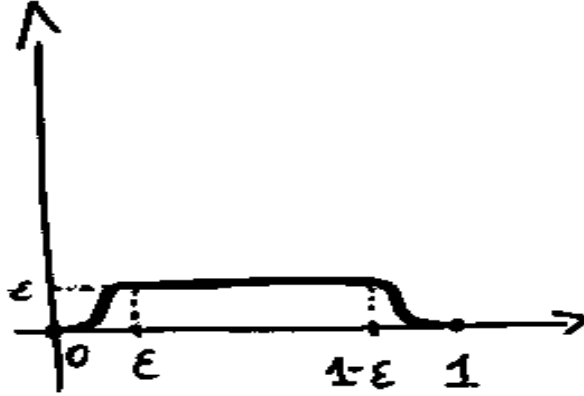


FIGURE 3. Bump function

*Remark 4.2.* We assume  $\epsilon$  to be sufficiently small as needed.

Consider a diagram

$$(\{(i, j) \mid i = 1, \dots, j, j = 1, \dots, n+1\}, \{(i_j, j) \mid j = 1, \dots, n+1\}, \text{tar})$$

corresponding to an element  $w \in S_{n+1}$ .

With each point  $(i, j)$  one associates a variable  $t_{i,j}$ . Later variables

$$\{t_{i,j} \mid i = 1, \dots, j; j = 1, \dots, n\}$$

will be variables of integration, while variables

$$t_{i,n+1}, \quad i = 1, \dots, n+1$$

will have the meaning of arguments. So let

$$\{t_{1,n+1}, t_{2,n+1}, \dots, t_{n+1,n+1}\}$$

be fixed and

$$0 < |t_{1,n+1}| < |t_{2,n+1}| < \dots < |t_{n+1,n+1}| \quad .$$

*NOTATION.* Variables  $t_{1,n+1}, \dots, t_{n+1,n+1}$  will be also denoted by  $z_1, \dots, z_{n+1}$ , correspondingly.

**Definition 4.3.** For each  $t_{i,j}$ ,  $i = 1, \dots, j$ ;  $j = 1, \dots, n$  define a contour  $t_{i,j}(\tau_{i,j})$ ,  $\tau_{i,j} \in [0, 1]$  as

$$t_{i,j}(\tau_{i,j}) = e^{2\pi i \tau_{i,j}} (1 - f_\epsilon(\tau_{i,j})) t_{tar(i,j)}.$$

See fig. 4a, 4b, 5a, 5b, 5c, 5d, 5e, 5f.

The described system of contours corresponding to  $w \in S_{n+1}$  will be denoted by  $\Delta_w = \Delta_w(z)$  and used as a cycle for integration of the form described in the next section. Compare also with [6,7,8,9,10,11,12,13,32].

*Remark 4.4.* In  $\Delta_w$  one has  $|t_{ij}| \leq |t_{tar(ij)}|$ , cf. the asymptotic zone of [14].

## 5. MULTIVALUED FORM $\omega_w$

Let

$$(\{(i,j) \mid i = 1, \dots, j, j = 1, \dots, n+1\}, \{(i_j, j) \mid j = 1, \dots, n+1\}, tar)$$

be a diagram corresponding to an element  $w \in S_{n+1}$ .

**Definition 5.1.** Define

$$x: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$$

to be the projection on the first factor, i.e.

$$x(a, b) = a.$$

**Definition 5.2.** Define  $\omega_w$  as

$$\begin{aligned} \omega_w := & \prod_{i=1}^{n+1} t_{i,n+1}^{\lambda_1 + \frac{kn}{2}} \prod_{i_1 \geq i_2} (t_{i_1,n+1} - t_{i_2,n+1})^{1-2k} \prod_{j=1}^n \left\{ \prod_{i_1 < x(tar(i,j))} (t_{ij} - t_{i_1,j+1})^{k-1} \right. \\ & \times \prod_{i_1 \geq x(tar(i,j))} (t_{i_1,j+1} - t_{ij})^{k-1} \prod_{i_1 > i_2} (t_{i_1,j} - t_{i_2,j})^{2-2k} \\ & \times \left. \prod_{i=1}^j t_{ij}^{\lambda_n - j + 2 - \lambda_n - j + 1 - k} \right\} dt_{11} dt_{12} dt_{22} \dots dt_{nn} \end{aligned}$$

cf. [1, 9, 10, 15, 17].

In  $\Delta_w$  we have  $t_{ij} = t_{ij}(\tau_{ij})$ ,  $i = 1, \dots, j$ ,  $j = 1, \dots, n$  ( $t_{i,n+1}$   $i = 1, \dots, n+1$  are fixed) and the phase of factors in the formula of  $\omega_w$  should be chosen so that it goes to zero as  $\tau_{ij}$  approach to zero, provided  $k$  and  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_{n+1}$  are real, cf. Remark 4.4.

*Remark 5.3.* Index  $w$  of the multivalued form above is, of course, a convention about the phase of the same form over the cycle  $\Delta_w$ .

## 6. MAIN THEOREMS

**Theorem 6.1.** *Let  $w \in S_{n+1}$ . Then the integral of the multivalued form  $\omega_w$  over cycle  $\Delta_w$  gives an asymptotic solution  $\phi(w\lambda + \rho, k, z)$ :*

$$\phi(w\lambda + \rho, k, z) = \int_{\Delta_w(z)} \omega_w = a(w) z^{w\lambda + \rho} (1 + \dots)$$

where

$$z^{w\lambda + \rho} = z_1^{\lambda_{w(1)} + \frac{kn}{2}} z_2^{\lambda_{w(2)} + \frac{k(n-2)}{2}} \dots z_{n+1}^{\lambda_{w(n+1)} - \frac{kn}{2}}$$

and

$$a(w) = \prod_{\alpha \in R_+} \frac{\Gamma((-w\lambda, \alpha^\vee)) \sin(\pi(-w\lambda, \alpha^\vee))}{\Gamma((-w\lambda, \alpha^\vee) + k)} \times e^{-2\pi i(\lambda, \delta)} e^{-\pi i(k-1)l(w)} \Gamma(k)^{\frac{n(n+1)}{2}} (2i)^{\frac{n(n+1)}{2}}.$$

The theorem is proved by induction on number of rows of a diagram.

**Mechanism of induction 6.2.** The mechanism of induction is based on the following simple observation. Let a diagram

$$(\{i, j\}, \quad \{i_j, j\}, \quad tar \mid i = 1, \dots, j, j = 1, \dots, n+1)$$

with  $n+1$  rows corresponds to an element  $w \in S_{n+1}$ .

Consider a diagram with  $n$  rows, which is obtained from the diagram with  $n+1$  rows by deleting the  $n+1$  row :

$$(\{i, j\}, \quad \{i_j, j\}, \quad tar \mid i = 1, \dots, j, j = 1, \dots, n)$$

and suppose that it corresponds to  $w' \in S_n$ . Then one has:

$$w(i) = \begin{cases} w'(i) + 1, & \text{if } i < i_{n+1} \\ w'(i-1) + 1, & \text{if } i > i_{n+1} \\ 1, & \text{if } i = i_{n+1} \end{cases}$$

cf. also theorems 2.10 and 2.6 and fig. 6.

**Theorem 6.3.** *The integrals of multivalued forms  $\omega_w$  over cycles  $\Delta_w$  satisfy to the following differential operator of second order:*

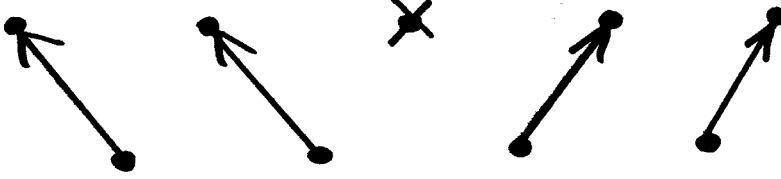


FIGURE 6. Illustration of induction

$$\left\{ \sum_{i=1}^{n+1} \left( z_i \frac{\partial}{\partial z_i} \right)^2 - k \sum_{i < j} \frac{z_j + z_i}{z_j - z_i} \left( z_i \frac{\partial}{\partial z_i} - z_j \frac{\partial}{\partial z_j} \right) \right\} \int_{\Delta_w} \omega_w =$$

$$((\lambda, \lambda) - (\rho, \rho)) \int_{\Delta_w} \omega_w$$

as well as to the whole hypergeometric system of differential equations and thus provide the basis for the solutions of this system for generic  $\lambda$  and  $k$ .

**Proof of theorem 6.3.** Throughout the proof we do not pay attention to the phases of multivalued forms, since they do not matter for the differential equations.

The theorem is proved as follows. Consider the following integral (which is a part of the multiple integral)

$$\int t_{11}^{\lambda_n - \lambda_{n-1} - k} (t_{12} - t_{11})^{k-1} (t_{22} - t_{11})^{k-1} dt_{11}.$$

Then it satisfies to the following differential equation

$$\left\{ t_{12} t_{22} \frac{\partial^2}{\partial t_{12} \partial t_{22}} + (k-1) \frac{t_{12} t_{22}}{t_{12} - t_{22}} \left( \frac{\partial}{\partial t_{12}} - \frac{\partial}{\partial t_{22}} \right) \right\} \int t_{11}^{\lambda_n - \lambda_{n-1} - k} (t_{12} - t_{11})^{k-1} (t_{22} - t_{11})^{k-1} dt_{11} = 0$$

Now let's integrate this differential operator by parts and rewrite it in terms of variables of the third row ( one should collect the terms and use homogeneity condition), i.e.  $t_{13}, t_{23}, t_{33}$ .

One obtains that the integral

$$\begin{aligned} & \int \int \int t_{11}^{\lambda_n - \lambda_{n-1} - k} (t_{12} t_{22})^{\lambda_{n-1} - \lambda_{n-2} - k} (t_{11} - t_{12})^{k-1} (t_{11} - t_{22})^{k-1} \\ & \quad \times (t_{12} - t_{22})^{2-2k} (t_{13} - t_{12})^{k-1} (t_{13} - t_{22})^{k-1} (t_{23} - t_{12})^{k-1} (t_{23} - t_{22})^{k-1} \\ & \quad \times (t_{33} - t_{12})^{k-1} (t_{33} - t_{22})^{k-1} dt_{11} dt_{12} dt_{22} \end{aligned}$$

satisfies to the following differential equation:

$$\begin{aligned} & \{t_{13} t_{23} \frac{\partial^2}{\partial t_{13} \partial t_{23}} + t_{13} t_{33} \frac{\partial^2}{\partial t_{13} \partial t_{33}} + t_{23} t_{33} \frac{\partial^2}{\partial t_{23} \partial t_{33}} \\ & + (k-1) \{ \frac{t_{13} t_{23}}{t_{13} - t_{23}} (\frac{\partial}{\partial t_{13}} - \frac{\partial}{\partial t_{23}}) + \frac{t_{23} t_{33}}{t_{23} - t_{33}} (\frac{\partial}{\partial t_{23}} - \frac{\partial}{\partial t_{33}}) + \frac{t_{13} t_{33}}{t_{13} - t_{33}} (\frac{\partial}{\partial t_{13}} - \frac{\partial}{\partial t_{33}}) \} \} \end{aligned}$$

$$\begin{aligned} & \int \int \int t_{11}^{\lambda_n - \lambda_{n-1} - k} (t_{12} t_{22})^{\lambda_{n-1} - \lambda_{n-2} - k} (t_{11} - t_{12})^{k-1} (t_{11} - t_{22})^{k-1} \\ & \quad \times (t_{12} - t_{22})^{2-2k} (t_{13} - t_{12})^{k-1} (t_{13} - t_{22})^{k-1} (t_{23} - t_{12})^{k-1} (t_{23} - t_{22})^{k-1} \\ & \quad \times (t_{33} - t_{12})^{k-1} (t_{33} - t_{22})^{k-1} dt_{11} dt_{12} dt_{22} = \end{aligned}$$

$$\begin{aligned} & a_3 \int \int \int t_{11}^{\lambda_n - \lambda_{n-1} - k} (t_{12} t_{22})^{\lambda_{n-1} - \lambda_{n-2} - k} (t_{11} - t_{12})^{k-1} (t_{11} - t_{22})^{k-1} \\ & \quad \times (t_{12} - t_{22})^{2-2k} (t_{13} - t_{12})^{k-1} (t_{13} - t_{22})^{k-1} (t_{23} - t_{12})^{k-1} (t_{23} - t_{22})^{k-1} \\ & \quad \times (t_{33} - t_{12})^{k-1} (t_{33} - t_{22})^{k-1} dt_{11} dt_{12} dt_{22} \end{aligned}$$

where  $a_3 = a_3(\lambda, k)$  is some constant which takes into account homogeneity relations. Since finally the eigenvalue will be determined using Harish-Chandra homomorphism one can avoid calculation of this constant.

Let's integrate by parts once more and rewrite differential operator in terms of  $t_{i4}$ ,  $i = 1, 2, 3, 4$  and one can do this row by row.

Calculation (integration by parts) amounts essentially to the following identity:

$$\sum_{i < j, s_1, s_2} \frac{t_{i,m} t_{j,m}}{(t_{s_1, m+1} - t_{i,m})(t_{s_2, m+1} - t_{j,m})} = \sum_{s_1, s_2, i < j} \frac{t_{s_1, m+1} t_{s_2, m+1}}{(t_{s_1, m+1} - t_{i,m})(t_{s_2, m+1} - t_{j,m})} + C(m)$$

where  $C(m) = C(m, \lambda, k)$  a constant which takes into account homogeneity relations.

On the last step the following lemma is used

**Lemma 6.4.**

$$\sum_{i < j} (t_{i,n+1} t_{j,n+1} \frac{\partial^2}{\partial t_{i,n+1} \partial t_{j,n+1}} + (k-1) \frac{t_{i,n+1} t_{j,n+1}}{t_{i,n+1} - t_{j,n+1}} (\frac{\partial}{\partial t_{i,n+1}} - \frac{\partial}{\partial t_{j,n+1}})) \prod_{p < q} (t_{p,n+1} - t_{q,n+1})^{2k-1} =$$

$$(2k-1) \frac{(n-1)n(n+1)}{24} (k6n - 3n + 2) \prod_{p < q} (t_{p,n+1} - t_{q,n+1})^{2k-1}$$

Lemma 6.4 easily follows from the next lemma 6.5.

**Lemma 6.5.**

$$\sum_{i < j} (t_{i,n+1} t_{j,n+1} \frac{\partial^2}{\partial t_{i,n+1} \partial t_{j,n+1}}) \prod_{p < q} (t_{p,n+1} - t_{q,n+1}) =$$

$$\frac{(n-1)n(n+1)}{24} (3n+2) \prod_{p < q} (t_{p,n+1} - t_{q,n+1})$$

**Proof of Lemma 6.5.**

In fact,

$$\prod_{p < q} (t_{p,n+1} - t_{q,n+1}) = \sum_{w \in S_{n+1}} (-1)^{\frac{n(n+1)}{2}} \det(w) \prod t_{w(i),n+1}^{i-1}$$

and lemma follows from the identity:

$$\sum_{q=2}^{n+1} \sum_{p < q} (p-1)(q-1) = \frac{(n-1)n(n+1)(3n+2)}{24}$$

**Continuation of proof of theorem 6.3.**

$$\left\{ \sum_{i=1}^{n+1} (t_{i,n+1} \frac{\partial}{\partial t_{i,n+1}})^2 - k \sum_{i < j} \frac{t_{j,n+1} + t_{i,n+1}}{t_{j,n+1} - t_{i,n+1}} (t_{i,n+1} \frac{\partial}{\partial t_{i,n+1}} - t_{j,n+1} \frac{\partial}{\partial t_{j,n+1}}) \right\} \int_{\Delta_w} \omega_w =$$

$$c_n \int_{\Delta_w} \omega_w$$

Finally, the eigenvalue can be determined using Harish-Chandra homomorphism

$$c_n = (w\lambda + \rho, w\lambda + \rho) - 2(\rho, w\lambda + \rho) = (\lambda, \lambda) - (\rho, \rho)$$

So

$$\left\{ \sum_{i=1}^{n+1} \left( t_{i,n+1} \frac{\partial}{\partial t_{i,n+1}} \right)^2 - k \sum_{i < j} \frac{t_{j,n+1} + t_{i,n+1}}{t_{j,n+1} - t_{i,n+1}} \left( t_{i,n+1} \frac{\partial}{\partial t_{i,n+1}} - t_{j,n+1} \frac{\partial}{\partial t_{j,n+1}} \right) \right\} \int_{\Delta_w} \omega_w = ((\lambda, \lambda) - (\rho, \rho)) \int_{\Delta_w} \omega_w$$

This completes the proof that the integrals satisfy to the second order differential equation and thus to the whole hypergeometric system of differential equations .

*Remark 6.6.* This elementary proof does not use the advanced theory of Knizhnik-Zamolodchikov equation (as well as Dunkl operators) ,but uses only integration by parts and the ability to present asymptotic solutions. Though one should notice that integration by parts is a commonly used technique in working with similar integrals appearing in the theory of Knizhnik-Zamolodchikov equation.

Let  $F_w$  be the normalized asymptotic solution of Heckman-Opdam hypergeometric system of type  $A_n$ , i.e.  $F_w(z) = z^{w\lambda+\rho}(1 + \dots)$ .

Now let  $z_1, z_2, \dots, z_{n+1}$  approach to 1 , while keeping inequalities

$$0 < |z_1| < |z_2| < \dots < |z_{n+1}|.$$

**Theorem 6.7.** (*Opdam*)

$$F_w(1) = \lim_{z \rightarrow 1} F_w(z) = \frac{\prod_{\alpha \in R_+} \frac{\Gamma((w\lambda, \alpha^\vee) + 1)}{\Gamma((w\lambda, \alpha^\vee) - k + 1)}}{\prod_{\alpha \in R_+} \frac{\Gamma(-(\rho, \alpha^\vee) + 1)}{\Gamma(-(\rho, \alpha^\vee) - k + 1)}}$$

*cf. theorem 6.3 [21].*

**Theorem 6.8.** *The limit of integral of  $\omega_w$  over  $\Delta_w$  as all  $z_i$  approach to 1 , while preserving the above inequalities, is equal to :*

$$\begin{aligned}
& \lim_{z \rightarrow 1} \int_{\Delta_w(z)} \omega_w = \\
& \prod_{\alpha \in R_+} \sin(\pi((-w\lambda, \alpha^\vee) + k)) \times e^{-2\pi i(\lambda, \delta)} e^{-\pi i(k-1)l(w)} (2i)^{\frac{n(n+1)}{2}} \\
& \frac{\sin(\pi k)^{n+1}}{\sin(\pi k) \sin(2\pi k) \dots \sin((n+1)\pi k)} \times \frac{\Gamma(k)^{\frac{(n+1)(n+2)}{2}}}{\Gamma(k) \Gamma(2k) \dots \Gamma((n+1)k)}
\end{aligned}$$

The theorem easily follows from theorems 6.1, 6.7, 6.6. see also [21, 17, 33].

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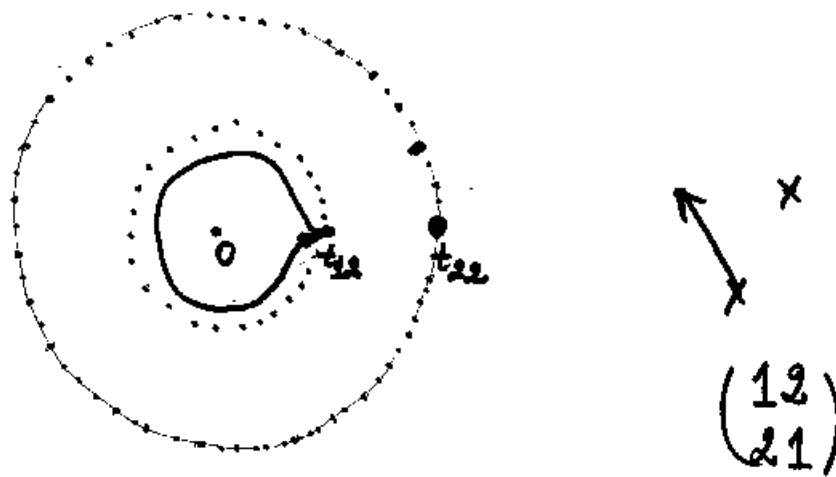
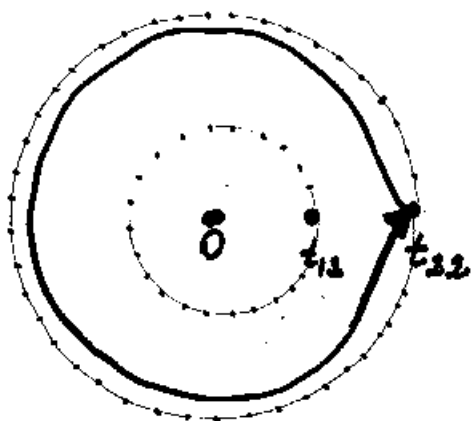


FIGURE 4A

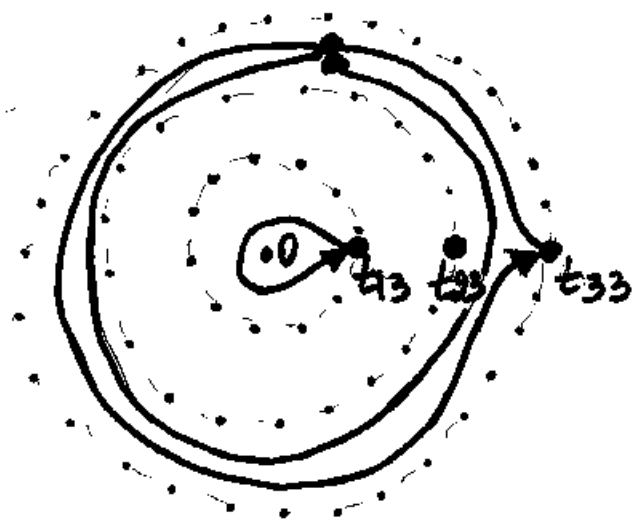
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$$\begin{array}{c} \times \\ \nearrow \\ \times \end{array}$$

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

FIGURE 4B



$$\begin{array}{c} \times \\ \nwarrow \quad \nearrow \\ \times \quad \times \end{array}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

FIGURE 5A

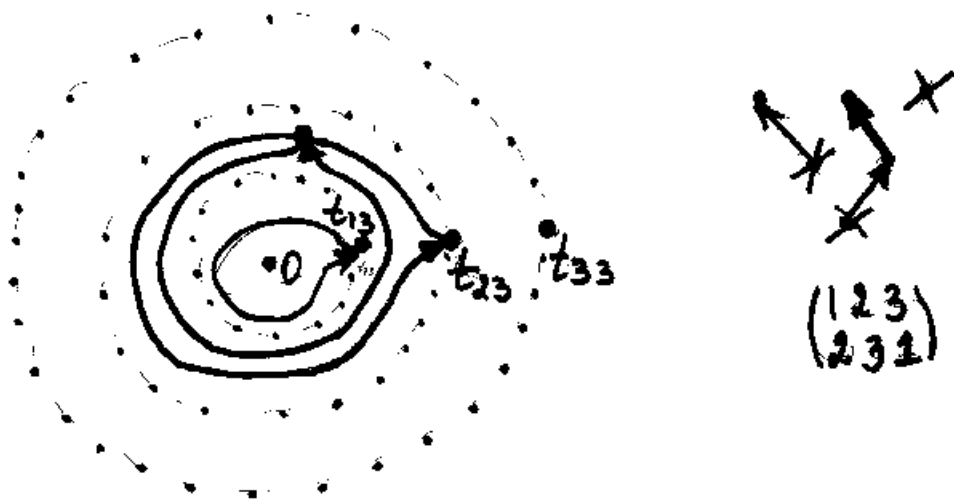


FIGURE 5B

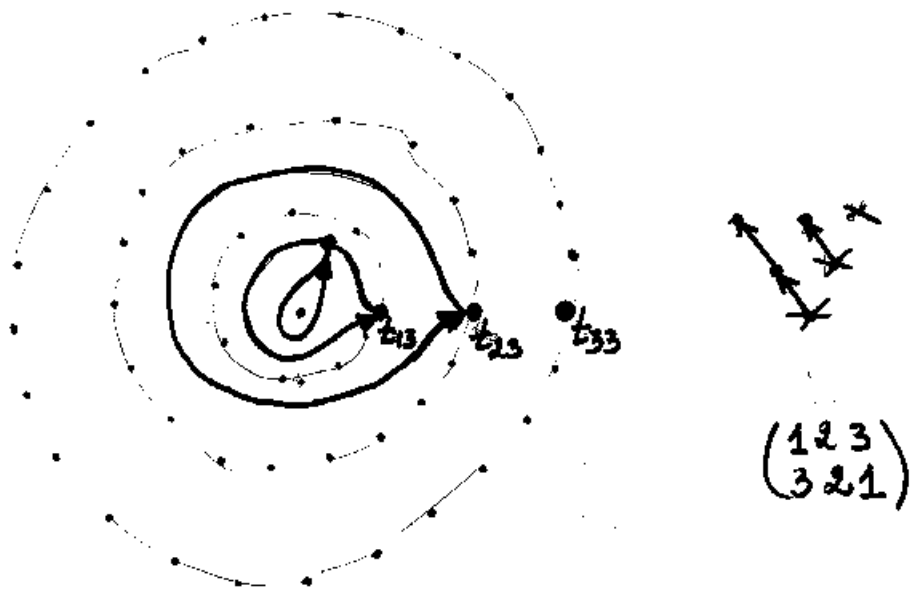


FIGURE 5C

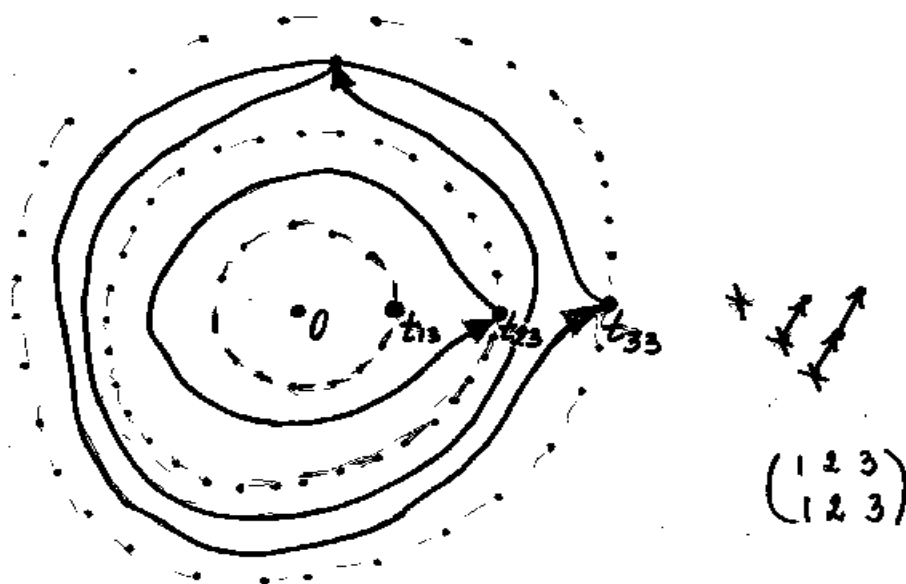


FIGURE 5D

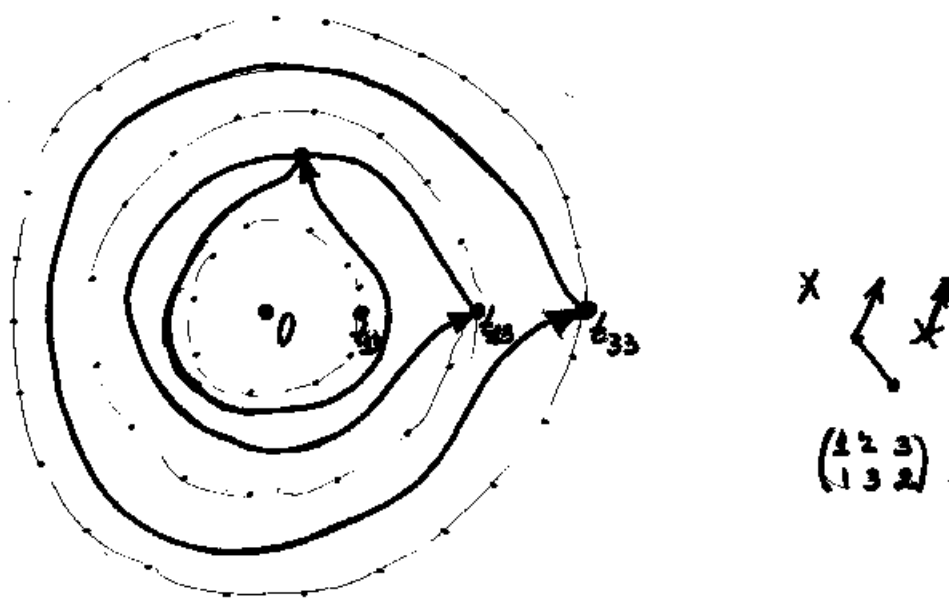


FIGURE 5E

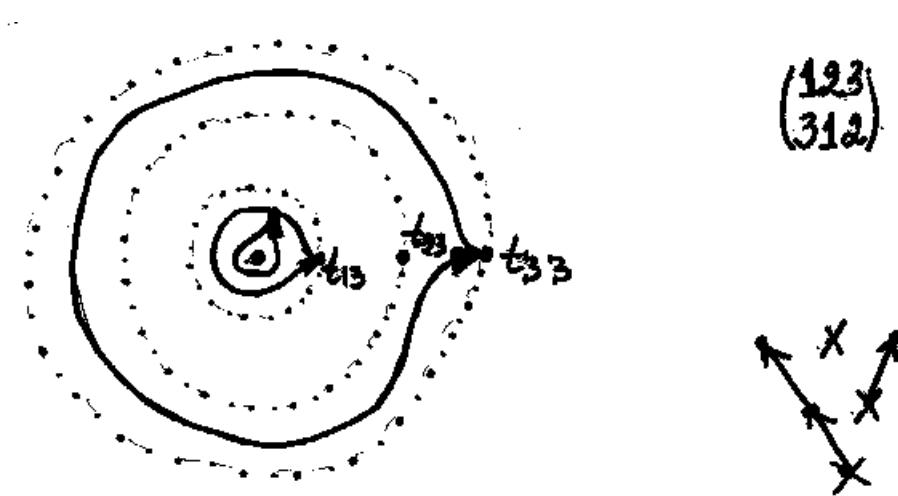


FIGURE 5F